MMA Sample Entrance Exam I Solutions

Problem 1. Number 12 can be written as a sum of an integer and its smallest divisor greater than 1 in two different ways: $12 = 10 + 2 = 9 + 3$. Find the smallest possible integer that can be represented as a sum of an integer and its smallest divisor greater than 1 in four different ways.

Solution. An integer n's smallest divisor d larger than 1 is its smallest prime divisor (if the smallest divisor were composite, it would have a prime divisor smaller than it, contradiction). Also, because n is a multiple of d, d is also a divisor of $n + d$. So, we are looking for a number with at least four distinct prime divisors, and the smallest such number is $210 = 2 \cdot 3 \cdot 5 \cdot 7$. It's easy to verify that 210 can indeed be represented as this sum in four ways: $208 + 2$, $207 + 3$, $205 + 5$, and $203 + 7$. So $\boxed{210}$ is the smallest possible such integer. □

Problem 2. Prove that for any positive integer n the following identity holds:

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n \cdot (n+1) \cdot (n+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Solution. We proceed by induction. It is clear that this holds for $n = 1$; suppose it holds for $n = k$. Then

$$\begin{aligned} &1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k \cdot (k+1) \cdot (k+2) + (k+1) \cdot (k+2) \cdot (k+3) \\ = &\frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\ = &\frac{(k+1)(k+2)(k+3)(k+4)}{4} \end{aligned}$$

so the identity also holds for $k + 1$. Thus, the identity holds for all positive integers n. □

Problem 3. The fraction $\frac{3}{4}$ is written on the board. Every minute, Fanny chooses two integer numbers. The first number which is always between 80 and 100 (inclusive) is added to the numerator, while the second number between 100 and 120 (also inclusive) is added to the denominator. If at any point the numerator and denominator have a common factor, it can be cancelled. Can Fanny eventually get $\frac{2}{3}$?

Solution. No. Let n be the number Fanny adds to the numerator and d the number she adds to the denominator. Notice that $\frac{n}{d} \geq \frac{2}{3}$ always. In general, given some fraction $\frac{p}{q} > \frac{2}{3}$, we see that $3p > 2q$ and $3n \geq 2d$, so $3(p + n) > 2(q + d)$ and $\frac{p+n}{q+d} > \frac{2}{3}$. In other words, if we start with a fraction greater than $\frac{2}{3}$ and repeatedly add these numbers to the numerator and denominator, we will still end up with a fraction greater than $\frac{2}{3}$. Because $\frac{3}{4} > \frac{2}{3}$, then, Fanny will never get $\frac{2}{3}$. □

Problem 4. For which values of parameter r does the equation

$$(r-3)x^2 - 2(r-2)x + r = 0$$

have 2 distinct real roots both greater than -1? Justify your answer.

Solution. Using the quadratic formula, the roots of this quadratic are

$$\frac{2(r-2) \pm \sqrt{4(r-2)^2 - 4r(r-3)}}{2(r-3)} = \frac{r-2 \pm \sqrt{-r+4}}{r-3}.$$

For the roots to be distinct and real, we must have $r < 4$. For them to be greater than -1, we want

$$\begin{aligned}
\frac{r-2 \pm \sqrt{-r+4}}{r-3} &> -1 \\
r-2-\sqrt{-r+4} &> -r+3 \\
2r-5 &> \sqrt{-r+4} \\
4r^2-20r+25 &> -r+4 \\
4r^2-19r+21 &> 0 \\
(4r-7)(r-3) &> 0
\end{aligned}$$

which occurs either when $r > 3, r > 7/4$ or $r < 3, r < 7/4$. So this equation has 2 distinct real roots both greater than -1 when $r \in \boxed{(-\infty, 7/4) \cup (3,4)}$. □

Problem 5. Eleven white chairs are placed around a circular table. They are numbered 1 through 11 in increasing order. In how many ways can Danielle paint some (or none) of the chairs red so that no three consecutive chairs are red?

Solution. We casework on the number of red chairs. Call a painting of the chairs good if it does not contain three consecutive chairs (and bad if it does).

If there are no red chairs, we automatically fulfill the conditions, for a total of 1 way.

If there is 1 red chair, it can be anywhere, for 11 ways.

If there are 2, they can also be anywhere, for $\binom{11}{2} = 55$ ways.

If there are 3, there are $\binom{11}{3}$ ways to choose chairs to paint, but 11 of those ways will have 3 consecutive chairs, so there are $\binom{11}{3} - 11 = 154$ good ways to paint them.

If there are 4, there are $\binom{11}{4}$ ways to choose chairs to paint. However, we could have 4 consecutive reds or groups of 3 and 1 red chair (separated by at least one white chair). The former can occur in 11 ways and the latter in $11 \cdot 6$ ways (after choosing the group of 3 chairs, the lone red chair can go in one of the 6 spots not adjacent to any of the other red chairs). So there are $\binom{11}{4} - 77 = 253$ good ways to paint 4 red chairs.

If there are 5, there are $\binom{11}{5}$ ways to choose chairs to paint. Then there are 3 types of bad ways, based on the largest group of consecutive red chairs (3, 4, or 5). If the largest group of consecutive red chairs consists of 3 chairs, then there are $11 \cdot 15$ bad ways to paint: 11 to choose the group of 3 chairs and $\binom{6}{2} = 15$ ways to choose where the remaining two chairs go. Likewise, if the largest group of red chairs consists of 4 chairs, there are $11 \cdot 5$ ways to paint, and if the largest group consists of 5 red chairs, there are 11 ways to paint them. So there are $\binom{11}{5} - 11 \cdot 21 = 231$ good ways to paint.

If there are 6 red chairs, there are $\binom{11}{6}$ ways to choose chairs to paint. Again, we count bad ways by doing casework on the largest group of consecutive red chairs. If the largest group of red chairs consists of 3 chairs, there are $11 \cdot 20$ bad ways: 11 to choose tho group of 3 chairs and $\binom{6}{3} = 20$ ways to choose where the remaining three chairs go. However, this overcounts when we have 2 groups of 3 chairs; there are $11 \cdot 2$ ways to paint chairs this way (11 locations for the first group, 4 for the second, and halved because the two groups are interchangeable). So in the 3-chair group case, there are $11 \cdot 18$ bad ways total. If the largest group has 4 chairs, there are $11 \cdot \binom{5}{2}$ bad ways; if the largest group as 5, there are $11 \cdot 4$ bad ways; and if the largest group has 6, there are 11 bad ways. So there are $\binom{11}{5} - 11 \cdot 33 = 99$ good ways to paint.

If there are 7 red chairs, there is exactly one way to paint the chairs up to rotation (3 pairs and one lone chair), so there are 11 good ways to paint.

If there are 8 or more red chairs, 3 of them must always be consecutive (we would need at least 12 chairs to have 4 pairs of 2 chairs).

Thus, in total there are $1 + 11 + 55 + 154 + 253 + 231 + 99 + 11$ good ways; this sums up to $\boxed{815}$ ways total. $\square$

Problem 6. Prove that the product of 100 consecutive positive integers can never be an exact 100th power of an integer.

Solution. Suppose this were possible. Let the product be p and its 100th root be k. First, k would have to be one of the 100 integers we are multiplying. Then we have two cases: either the 100 consecutive integers contain a multiple of 101, or not. Only the first case is possible: if they contain no multiple of 101, because 101 is prime, by Fermat's Little Theorem their product must be $1 \pmod{101}$. At the same time, their product is $100! \pmod{101}$ but by Wilson's Theorem, that is $-1 \pmod{101}$, contradiction. So exactly one of the 100 integers is a multiple of 101. Moreover, this integer must be k, since otherwise p would not be a multiple of 101. Let e be the largest integer so that $101^e | k$ with $e \geq 1$; however, all the powers of 101 in p must come from k and $101^{100e} | p$, so $101^{100e} | k$ also, contradiction. $\square$