MMA Sample Entrance Exam L Solutions

Problem 1. Place numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 around a circle so that sum of any two numbers next to each other is not divisible by 3, 5, or 7.

Solution. For each number, we can check which numbers it can be adjacent to:

1	3, 7
2	6, 9
3	1, 5, 8
4	7, 9
5	3, 6, 8
6	2, 5, 7
7	1, 4, 6
8	3, 5, 9
9	2, 4, 8

Because the numbers are around a circle, every number is adjacent to exactly two others. Then, we see that 1 must be next to 3 and 7; that 4 must be next to 7 and 9; and that 2 must be next to 9 and 6. Then the numbers next to 6 must be 2 and 5, and the numbers next to 5 are 8 and 6. So one way to place the numbers is 1, 7, 4, 9, 2, 6, 5, 8, 3 clockwise around the circle, and it is easy to verify that this works. $\square$

Problem 2. How many two-digit numbers give a perfect square when added to its "mirror" (the two-digit number written with its digits in reverse order)?

Solution. Represent a two-digit number by $\overline{AB}$ with A and B digits from 1 to 9 (inclusive) and $\overline{AB} = 10A + B$. Then the mirror of this number is $\overline{BA} = 10B + A$ and their sum is $11A + 11B$, or $11(A + B)$. For this to be a perfect square, $A + B = 11 \cdot n^2$ for some integer n, but because A, B are digits, $n = 1$. There are then $\boxed{8}$ pairs of digits which satisfy this, corresponding to 29, 38, 47, 56, 65, 74, 83, and 92. $\square$

Problem 3. A carpenter went to the store and bought 10 planks of wood. Each plank has a length that is a whole number of centimeters. The longest plank has a length of exactly 54 centimeters. Prove that there exist three planks that can be arranged to form a triangle.

Solution. Suppose there do not exist three planks that can be arranged to form a triangle. Consider the minimal lengths of shortest, second shortest, etc. planks. The shortest plank has minimal length 1, as does the second shortest. The next-shortest plank has minimal length 2 (otherwise, we could make a triangle out of 3 planks all of length 1). Likewise, the next-shortest plank has minimal length $3 = 2 + 1$, and the next-shortest has minimal length $5 = 3 + 2$, and so on. Repeating this (or noticing that the planks have lengths the Fibonacci numbers), the 10th plank has minimal length 55, but the longest plank is 54 centimeters, contradiction. $\square$

Problem 4. Number 12 can be written as a sum of an integer and its smallest divisor greater than 1 in two different ways: $12 = 10 + 2 = 9 + 3$. Find the smallest possible integer that can be represented as a sum of an integer and its smallest divisor greater than 1 in four different ways.

Solution. An integer n's smallest divisor d larger than 1 is its smallest prime divisor (if the smallest divisor were composite, it would have a prime divisor smaller than it, contradiction). Also, because n is a multiple of d, d is also a divisor of $n + d$. So, we are looking for a number with at least four distinct prime divisors, and the smallest such number is $210 = 2 \cdot 3 \cdot 5 \cdot 7$. It's easy to verify that 210 can indeed be represented as this sum in four ways: $208 + 2$, $207 + 3$, $205 + 5$, and $203 + 7$. So $\boxed{210}$ is the smallest possible such integer. □

Problem 5. Prove that for any positive integer n the following identity holds:

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n \cdot (n+1) \cdot (n+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Solution. We proceed by induction. It is clear that this holds for $n = 1$; suppose it holds for $n = k$. Then

$$\begin{aligned} & 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k \cdot (k+1) \cdot (k+2) + (k+1) \cdot (k+2) \cdot (k+3) \\ = & \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\ = & \frac{(k+1)(k+2)(k+3)(k+4)}{4} \end{aligned}$$

so the identity also holds for $k + 1$. Thus, the identity holds for all positive integers n. □

Problem 6. The fraction $\frac{3}{4}$ is written on the board. Every minute, Fanny chooses two integer numbers. The first number which is always between 80 and 100 (inclusive) is added to the numerator, while the second number between 100 and 120 (also inclusive) is added to the denominator. If at any point the numerator and denominator have a common factor, it can be cancelled. Can Fanny eventually get $\frac{2}{3}$?

Solution. No. Let n be the number Fanny adds to the numerator and d the number she adds to the denominator. Notice that $\frac{n}{d} \geq \frac{2}{3}$ always. In general, given some fraction $\frac{p}{q} > \frac{2}{3}$, we see that $3p > 2q$ and $3n \geq 2d$, so $3(p + n) > 2(q + d)$ and $\frac{p+n}{q+d} > \frac{2}{3}$. In other words, if we start with a fraction greater than $\frac{2}{3}$ and repeatedly add these numbers to the numerator and denominator, we will still end up with a fraction greater than $\frac{2}{3}$. Because $\frac{3}{4} > \frac{2}{3}$, then, Fanny will never get $\frac{2}{3}$. □