Group Y Math-M-Addicts Entrance Exam 2014 – 2015

Student Name ____Grade in School ___________________

School Name _____________________________________Email ______________________________

Instructions: A perfect solution to each problem below is worth 10 points. A score of 40 or more will guarantee admission to group Y, though depending on the submitted tests, the cutoff for admission may be lower. **Students must work on the problems on their own and not receive any help; students suspected of cheating will be disqualified.** Good luck!

1. Using numbers 5, 6, 7, 8 and basic arithmetic operations (i.e., +, -, *, /, and parentheses), generate 24. Each number has to be used exactly once and combining numbers (e.g., 78 - 56) is not allowed.

2. Find all solutions to TAE + TEA = EAT; each letter corresponds to a digit; same letter corresponds to the same digit and different letters correspond to different digits.

3. Positive numbers p and $p^2 + 7$ are both prime. What are all possible values of p? Justify your answer.

4. A scientist gathered four math students. They were lined up so that each one could see the ones in front of them but not behind them. Each student had a hat placed on their head. So the student in the back could see the hats of the three students in front, but the student in front could not see any hats. *"There is a red hat, a white hat, a blue hat, and a hat that is a duplicate of one of those colors,"* the scientist said. Starting with the one in the back, each student was asked what color hat they were wearing. They were all able to give correct answers without guessing! Which two students were wearing the two hats of the same color? Explain your reasoning.

5. If 1 + 2 + 3 + 4 + ... + 300 = 45,150, compute 2 + 4 + 6 + 8 + ... + 300 without using a calculator. Explain your reasoning.

6. All numbers between 1 and 999 were written out into one very large number. How many 7's were used? Explain your reasoning.

Math-M-Addicts Entrance Exam Solutions
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Problem 1. Using numbers 5, 6, 7, 8 and basic arithmetic operations (i.e., +, -, *, /, and parentheses), generate 24. Each number has to be used exactly once and combining numbers (e.g., 78 - 56) is not allowed.

Solution: We note that 24 = 12 * 2 = 8 * 3 = 6 * 4, which is reflected in our solutions: $\boxed{24 = 6 * (7 + 5 - 8)}$, $\boxed{24 = 8 * 6 / (7 - 5)}$, $\boxed{24 = (7 + 5) * (8 - 6)}$.

Problem 2. Find all solutions to TAE + TEA = EAT; each letter corresponds to a digit; same letter corresponds to
the same digit and different letters correspond to different digits.

Solution 1: Consider the addition of the middle digits: A + E = A. It can only be possible if E = 0 and there was a carry of 0 or if E = 9 and there was a carry of 1 (since we are adding just two numbers, only carries of 0 and 1 are possible). Since EAT starts with E, E cannot be 0, therefore E = 9. Therefore, from looking at leading digits, T = 4, which leads to A = 5 from looking at the last digit. Answer: $\boxed{T = 4, A = 5, \text{ and } E = 9}$.

Solution 2: The statement of the problem is equivalent to

$$100T + 10A + E + 100T + 10E + A = 100E + 10A + T,$$

which simplifies to

$$199T + A = 89E$$

Since TAE + TEA is a 3-digit number, T = 1, 2, 3, or 4. Plugging in, we find that T = 1, T = 2, and T = 3 yield no solutions because 89E cannot come close enough to 199T, and T = 4 yields the solution $\boxed{T = 4, A = 5, \text{ and } E = 9}$.

Problem 3. Positive numbers p and $p^2 + 7$ are both prime. What are all possible values of p? Justify your answer.

Solution: We note that p cannot be odd. Indeed, if p is odd, then $p^2 + 7$ is even; since the only even prime is 2, we would need $p^2 = -5$, which is impossible. Therefore, p is even, which means p =2. Checking, $p^2 + 7 = 11$, which is prime. Answer: $\boxed{p = 2}$.

Problem 4. A scientist gathered four math students. They were lined up so that each one could see the ones in front of them but not behind them. Each student had a hat placed on their head. So the student in the back could see the hats of the three students in front, but the student in front could not see any hats. *"There is a red hat, a white hat, a blue hat, and a hat that is a duplicate of one of those colors,"* the scientist said. Starting with the one in the back, each student was asked what color hat they were wearing. They were all able to give correct answers without guessing! Which two students were wearing the two hats of the same color? Explain your reasoning.

Solution: The last student must have seen two hats of the same color to guess his color correctly for if
all three hats in front of him were different colors, there would be no way for him to determine the color of his hat. Now consider the next-to-last student. If the hats in front of him are of different colors, he does not know whether his color matches the color of the first or the second hat. Therefore, the hats in front of him also need to be of the same color. The last two students will know that their hats are of the same, unmentioned, color. Hence, the two students whose hats are the same color are the first two.

<u>Problem 5</u>. If 1 + 2 + 3 + 4 + ... + 300 = 45,150, compute 2 + 4 + 6 + 8 + ... + 300 without using a calculator. Explain your reasoning.

Solution 1: Let 2 + 4 + 6 + 8 + ... + 300 = X. Then,

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots + 300 = 45{,}150$$

$$(2-1) + 2 + (4-1) + 4 + (6-1) + 6 + \cdots + (300-1) + 300 = 45{,}150$$

Rearranging the terms, we get:

$$(2 + 4 + 6 + \cdots + 300) + (2 + 4 + 6 + \cdots + 300) - 1 - 1 - 1 \ldots - 1 = 45{,}150$$

Noting that there are exactly 150 -1's and recalling our definition of X, we obtain:

$$2X - 150 = 45{,}150$$

$$X = 22{,}650$$

Answer: $\boxed{22{,}650}$.

Solution 2: We ignore the help we are given and compute 2 + 4 + 6 + 8 + ... + 300 directly. Matching up the first number with the last number, the second number with the next-to-last number, etc., we get 302 + 302 + … + 302, 75 times since we have 150 numbers. Answer = 302 * 75 = $\boxed{22{,}650}$. The answer makes sense since it is about half of 45,150.

<u>Problem 6</u>. All numbers between 1 and 999 were written out into one very large number. How many 7's were used? Explain your reasoning.

Solution 1: We will go ahead and count directly. The last digit is 7 in 7, 17, 27, …, 997 – one number in ten, which is 100 numbers. The first digit is 7 in 700 – 799 – also 100 numbers. Now, for the middle 7. It is in 10 numbers between 0 and 100 (70 – 79), in 10 numbers between 100 and 199 (170 – 179), etc., up until occurring 10 times as the middle digit between 900 and 999 (970 – 979), for a grand total of also 100. Answer: 100 + 100 + 100 = $\boxed{300}$.

Solution 2: Let's change the problem – instead of writing numbers from 1 to 999, we write numbers from 0 to 999. Clearly the answer does not change. Now, let's change the problem again – instead of writing out a number, we will also write out the leading 0's so as to complete a 3-digit number. That is, we will write 000, 001, 002, 003, …, 010, 011, …, 099, 100, 101, …, 999. Now, note that we wrote every combination of 3 digits exactly once! So each digit is repeated the same number of times, by symmetry. Since there are 1,000 numbers, we wrote 3,000 digits, and therefore we wrote exactly $\boxed{300}$ 7's (as well as 300 1's, 300 2's, etc.).